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17/5/18

17/5

This question paper contains 3 printed pages.

Roll No.

S. No. of Question Paper:

5720

Unique Paper Code

: 235401

Name of the paper

: MAHT - 401: Differential Equations and Mathematical Modelling II

Name of the course

: B.Sc. (Hons.) Mathematics ~~Part II~~

Semester

: IV

Duration: 3 Hours

Maximum Marks: 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All questions are compulsory.

Section I

1. Attempt any ^{parts} two questions out of the following:

(a) Prove that the general solution of the linear equation

$$x(y-z)u_x + y(z-x)u_y + z(x-y)u_z = 0, \text{ is } u(x, y, z) = f(x+y+z, xyz) \quad (7\frac{1}{2})$$

(b) Solve the Cauchy problem

$$u(x+y)u_x + u(x-y)u_y = x^2 - y^2, \text{ with the Cauchy data } u = 0 \text{ on } y = 2x. \quad (7\frac{1}{2})$$

(c) Apply the method of separation of variables to solve the equation ~~$u(x, y) = f(x)g(y)$~~

~~to solve the problem~~ $u_x + u = u_y, u(x, 0) = 4 \exp(-3x). \quad (7\frac{1}{2})$

Section II

2. Attempt any ^{part} one question out of the following:

(a) Show that the equation of motion of a long string is $u_{tt} = u_{xx} - g$, where g is the gravitational acceleration. (6)

(b) Reduce the equation $u_x + xu_y = y$ to canonical form, and obtain the general solution. (6)



two parts
3. Attempt any ~~two~~ ^{three} questions out of the following:

(a) Obtain the general solution of the equation

$$u_{xx} + u_{xy} - 2u_{yy} - 3u_x - 6u_y = 9(2x - y), \quad (6)$$

(b) Transform the equation $3u_{xx} + 7u_{xy} + 2u_{yy} + u_x + u = 0$, to the form $v_{\xi\eta} = cv$, $c = \text{constant}$ by introducing the new variable $v = ue^{-(a\xi + b\eta)}$, where a and b are undetermined coefficients. (6)

(c) Obtain the general solution of the equation $u_{xxx} + 2u_{xxy} + u_{yyy} = 0$. (6)

Section III

part 3
4. Attempt any ~~three~~ ^{three} out of the following:

(a) Obtain the solution of the following Cauchy problem

$$u_{tt} = c^2 u_{xx} + x + ct$$

with initial conditions

$$u(x, 0) = x, \quad u_t(x, 0) = \sin x.$$

(b) State and derive celebrated D'Alembert solution of the Cauchy problem for the one dimensional wave equation. (7)

(c) Solve the initial value problem:

$$u_{tt} = c^2 u_{xx}, \quad 0 \leq x \leq \pi, t > 0$$

$$u(x, 0) = \sin x, \quad u_t(x, 0) = x^2 - \pi x, \quad 0 \leq x \leq \pi$$

$$u(0, t) = u(\pi, t) = 0, \quad t > 0. \quad (7)$$

(d) Solve the problem:

$$xy^3 u_{xxx} - x^3 y u_{yyy} - y^3 u_{xx} + x^3 u_{yy} = 0,$$

$$u(x, y) = f(x) \text{ on } y^2 + x^2 = 16 \text{ for } 0 \leq x \leq 4$$

$$u(x, y) = g(y) \text{ on } x = 0 \text{ for } 0 \leq y \leq 4$$

where $f(0) = g(4)$ (7)

Section IV

part 4
5. Attempt any ~~three~~ ^{three} questions out of the following:

(a) A string of constant tension T and density ρ with $c^2 = T/\rho$ stretched along the x -axis from 0 to l and is fixed at its end points. The string is raised to a height h at $x = a$ and then released. Determine the equation of the displacement of this plucked string. (7)

(b) Prove the uniqueness of the solution of the problem:

$$u_t = k u_{xx}, \quad 0 < x < l, \quad t > 0,$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq l,$$

$$u(0, t) = 0, \quad u(l, t) = 0, \quad t \geq 0$$

(c) Determine the solution of the initial value problem

$$u_{tt} = c^2 u_{xx} + x^2, \quad 0 < x < l, \quad t > 0,$$

$$u(x, 0) = x, \quad 0 \leq x \leq l$$

$$u_t(x, 0) = 0, \quad 0 \leq x \leq l,$$

$$u(0, t) = 0, \quad u(l, t) = 1, \quad t \geq 0.$$

(d) By using the method of separation of variables, solve the telegraph equation

$$u_{tt} + a u_t + b u = c^2 u_{xx}, \quad 0 < x < l, \quad t > 0,$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq l,$$

$$u_t(x, 0) = 0, \quad 0 \leq x \leq l,$$

$$u(0, t) = 0, \quad u(l, t) = 0, \quad t \geq 0.$$



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Sr. No. of Question Paper

: 8721

Roll No.

Unique Paper Code

: 235402

Name of the Course

: B.Sc. (II) Mathematics (Semester-Mode)

Name of the Paper

: IV.2 MAHT 402-Analysis-III

Semester

: IV

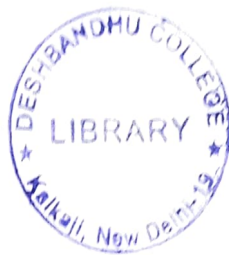
Duration

: 3 Hours

Maximum Marks:75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any two parts from each question.
3. All questions are compulsory.



1. (a) Define Upper and Lower Darboux sums of a bounded function f on $[a, b]$. Show that a bounded function f on $[a, b]$ is integrable iff for each $\epsilon > 0$ there exists a partition P such that $U(f, P) - L(f, P) < \epsilon$ (6)
(b) Find the upper and lower Darboux integrals for $f(x) = x^2$ on the interval $[0, b], b > 0$. (6)
(c) Show that every monotonic function f on $[a, b]$ is integrable. (6)
2. (a) Let f be a function on $[a, b]$. If $a < c < b$ and f is integrable on $[a, c]$ and on $[c, b]$, then f is integrable on $[a, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$. (6)
(b) Give an example of a function f on $[0, 1]$ that is not integrable for which $|f|$ is integrable. (6)
Justify your answer.
(c) Suppose f and g are continuous functions on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists x in $[a, b]$ such that $f(x) = g(x)$. (6)
3. (a) Discuss the convergence of the following improper integrals:

(i) $\int_0^{\infty} \frac{2^t + t}{3^t + t} dt$

(ii) $\int_1^{\infty} \sin \frac{1}{t} dt$

(3+3)

(b) Let $p, q \in \mathbb{R}, p > 0, q > 0$. Show that

$$\beta(p, q) = 2 \int_0^{\frac{\pi}{2}} (\sin u)^{2p-1} (\cos u)^{2q-1} du \quad (6)$$

(c) Define Gamma function. Show that

(i) $\Gamma(s) > 0$ for all $s > 0$.

(ii) $\Gamma(s+1) = s\Gamma(s)$ for all $s > 0$.

(iii) $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$. (6)

4. (a) Let $f_n(x) = \frac{x^n}{1+x^n}$ for $x \geq 0$. Show that the sequence $\langle f_n \rangle$ converges only pointwise on $[0, 1]$ and converges uniformly on $[0, b]$, $0 < b < 1$. (7)

(b) Show that the sequence $f_n(x) = x + \frac{1}{n}$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$, converges uniformly on $\mathbb{R}$ but the sequence $\langle f_n^2 \rangle$ does not converge uniformly on $\mathbb{R}$. (7)

(c) Let $\langle f_n \rangle$ be a sequence of functions on $R[a, b]$ and suppose that $\langle f_n \rangle$ converges uniformly on $[a, b]$ to f . Then show that $f \in R[a, b]$ and $\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$. (7)

5. (a) Let $\langle f_n \rangle$ be a sequence of bounded functions such that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, $x \in [a, b]$ and let

$\|f_n - f\| = \sup_{x \in [a, b]} |f_n(x) - f(x)|$. Then $\langle f_n \rangle$ converges to f uniformly on $[a, b]$ if and only if $\|f_n - f\| \rightarrow 0$ as $n \rightarrow \infty$. (6.5)

(b) Show that the series $\sum \frac{1}{n^2 + x^2}$ is uniformly convergent on $\mathbb{R}$. (6.5)

(c) Define Exponential function E on $\mathbb{R}$. Further show that

(i) E is strictly increasing on $\mathbb{R}$.

(ii) $E(x+y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$. (6.5)

Q4(b) Let $f_n(x) = x + \frac{1}{n}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. Show that the sequence $\langle f_n \rangle$ converges uniformly on $\mathbb{R}$ but the sequence $\langle f_n^2 \rangle$ does not converge uniformly on $\mathbb{R}$.

6.(a) Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then prove that

$$\int_0^x f(t) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \text{ for } |x| < R. \quad (6)$$

(b) Show that $\log(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n$ for $|x| < 1$ and hence establish the identity

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \quad (6)$$

(c) Prove that $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$, and evaluate ^{Evaluate}

(i) $\sum_{n=1}^{\infty} \frac{n}{3^n}$

(ii) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$

(6)



17/5/14
1913
Page No. 222
Page Code: 238424

Name of the Paper: MAHT 403: Algebra-III

Name of the Course: B.Sc. (H) Mathematics

Semester: IV

Duration: 3 hrs

Maximum marks: 75



Instructions for Candidates: Attempt any two parts from each question.

Q1. (a) Let R be a commutative ring with unity and let $U(R)$ denote the set of units of R . Prove that $U(R)$ is a group under multiplication of R .

(b) (i) Does the unity of a subring have to be the same as unity of the whole ring? If yes, prove it. If not give an example.

(ii) Give an example of an ideal which is a prime ideal but not a maximal ideal.

(c) Prove that a non-empty subset S of a ring R is a subring if S is closed under subtraction and multiplication that is, if $a - b$ and ab are in S whenever a and b are in S .

(6,6,6)

Q2. (a) Let R be a commutative ring with unity and let A be an ideal of R . Then R/A is an integral domain if and only if A is prime ideal.

(b) Prove that ideal $\langle x^2 + 1 \rangle$ is maximal in $R[x]$.

(c) Prove that $I = \langle 2 + 2i \rangle$ is not a prime ideal of $\mathbb{Z}[i]$. How many elements are in $\mathbb{Z}[i]/I$?

(6,6,6)

Q3. (a) Determine all ring homomorphisms from $\mathbb{Z}_8$ to $\mathbb{Z}_6$.

(b) Let ϕ be a ring homomorphism from R to S . Then the mapping from $R/\text{Ker } \phi$ to $\phi(R)$ given by $r + \text{Ker } \phi \mapsto \phi(r)$ is an isomorphism.

(c) Show that a homomorphism from a field onto a ring with more than one element must be an isomorphism.

(6,6,6)

Q4. (a) Let F be a field and $p(x)$ be an irreducible polynomial over F . Then prove that $\frac{F[x]}{\langle p(x) \rangle}$ is a field.

- (b) Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \in \mathbb{Z}[x]$. If there is a prime p such that $p \nmid a_n, p \nmid a_{n-1}, \dots, p \nmid a_0$ and $p^2 \nmid a_n$, then prove that $f(x)$ is irreducible over $\mathbb{Q}$.
- (c) Prove that in a principal ideal domain, an element is irreducible if and only if it is prime.

(6.5, 6.5, 6.5)

Q5. (a) Let W_1 and W_2 be subspaces of a vector space V .

- Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2 .
- Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

(b) Let V be a vector space and $\beta = \{u_1, u_2, \dots, u_n\}$ be a subset of V . Then prove that β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β .

(c) Let $S = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ be a subset of the vector space F^3 where F is a field.

- Prove that if $F = \mathbb{R}$ (field of real numbers), then S is linearly independent.
- Prove that if F has characteristic 2, then S is linearly dependent.

(6.5, 6.5, 6.5)

Q6. (a) Let V and W be vector spaces and $T: V \rightarrow W$ be a linear transformation. Then prove that $N(T)$ and $R(T)$ are subspaces of V and W respectively.

(b) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and $U: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformations defined by

$$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$$

$$U(a_1, a_2) = (a_1 - a_2, 2a_1, 3a_1 + 2a_2)$$

Show that $[T + U]_{\beta}^{\gamma} = [T]_{\beta}^{\gamma} + [U]_{\beta}^{\gamma}$, where β and γ are standard ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

(c) Suppose that $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is linear and $T(1, 0) = (1, 4)$, $T(1, 1) = (2, 5)$.

(6.5, 6.5, 6.5)

What is $T(2, 3)$? Is T one-to-one, explain?

Sl. No. of Q.P.: 5866(19)
Unique Paper Code : 222481

Name of the Paper : Physics – II

Name of the Course : B.Sc. (Hons) Mathematics

Semester : IV-semester

Duration : 3 hours

Maximum Marks : 75 Marks



Instructions for candidates:

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt **five** questions in all.

Question number 1 is compulsory.

1. Attempt any **five** questions .

5X3

- What is the peak inverse voltage (PIV) of a diode in a rectifier? Why is full-wave bridge rectifier preferred over a centre-tapped full wave rectifier?
- Define current gain α and current gain β of a transistor. Derive the relation between them.
- Define DC load line of a transistor. How is it obtained?
- Discuss Barkhausen criterion for self-sustaining oscillations in transistor oscillators.
- What are polar and non-polar molecules?
- Distinguish between dia, para- and ferro- magnetic materials.
- Derive the equation of continuity $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$ from the Maxwell's equation, where the symbols have their usual meaning.
- State De- Morgan's theorems and simplify the following expression:
 $(\overline{ABC}).(\overline{ABC})$

2.

- What is an electric dipole. Deduce the expressions for its electric field at an axial point and an equatorial point. Obtain the ratio of these fields. 11
- State and prove Gauss law in electrostatics. 4

3.

- Explain hysteresis in magnetic materials. Obtain an expression for energy loss during the magnetising cycle. 8

- b) Explain the laws of electromagnetic induction. Explain how Lenz's law gives direction of the induced emf in the circuit. 7

4. a) Describe the path of a charged particle moving in a uniform magnetic field and also show that there is no gain in energy in the process. 2
b) Show that the force per unit length between two parallel wires carrying currents I_1 and I_2 separated by distance d is given by $f = \frac{\mu_0 I_1 I_2}{2\pi d}$ 7

5. a) Draw the circuit diagram of a half-wave rectifier. Explain its working. Draw the corresponding input and the output waveforms. What is the frequency of ripple in its output? Derive the expression for the ripple factor of a half wave rectifier. 10
b) Why do we need filters in a power supply? Discuss the dependence of load resistance on ripple factor in the case of shunt capacitor, series inductor, LC section and π filter. 5

~~c) Why is Zener diode used as a voltage regulator?~~ 2

6. a) What do you understand by negative and positive feedback in an amplifier? Discuss the importance of negative feedback in an amplifier. 7
b) Discuss the hybrid equivalent circuit of a transistor in a two port network. Write the expressions for various hybrid parameters. 8
c) ~~Explain the working of a potential divider/self bias circuit for a transistor. Derive an expression for the stability factor.~~ 5

7. a) Draw AND, OR and NOT logic gates in terms of universal gate. 9
~~b) Simplify the following Boolean expression:
 $AB + A(B+C) + B(B+C)$~~ 6
c) Convert 1110111.0111 into its decimal, octal and hexa-decimal equivalent. 3
~~d) Represent -36 into its 2's complement form.~~ 3
~~e) How does a half adder circuit differ from a full adder circuit? Write their respective truth tables.~~ 3



20
This question paper contains 4 printed pages.

Your Roll No.

S. No. of Paper : 6632
Unique paper code : 32351401
Name of the paper : Partial Differential Equations
Name of course : B.Sc. (Hons.) Mathematics
Semester : IV
Duration : 3 hours
Maximum marks : 75

HC



(Write your Roll No. on the top immediately
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All Sections are compulsory.
Marks of each part are indicated.

SECTION - I

Attempt any two parts out of the following.

1. (a) Obtain the general solution of the equation

$$(z^2 - 2yz - y^2)p + x(y + z)q = x(y - z).$$

Find the integral surfaces of this equation passing through (i)
the x -axis, (ii) the y -axis, (iii) the z -axis.

$$\left(7\frac{1}{2}\right)$$

- (b) Solve the following initial value system:

$$u_t + uu_x = e^{-x}v, v_t - av_x = 0$$

with $u(x, 0) = x$ and $v(x, 0) = e^x$.

$$\left(7\frac{1}{2}\right)$$

P. T. O.

- (c) Apply the method of separation of variables $u(x, y) = f(x)g(y)$ to solve

$$x^2 u_{xx} + 2xy^2 u = 0 \quad u(x, 0) = \exp\left(\frac{1}{x}\right) \quad \left(7 \frac{1}{2}\right)$$

SECTION - II

Attempt any **one** part out of the following.

2. (a) Show that the equation of motion of a one-dimensional wave equation is $u_{tt} = c^2 u_{xx}$ where $c^2 = \tau/\rho$. (6)
- (b) Derive the Laplace equation of motion. (6)

Attempt any **two** parts out of the following.

3. (a) Find the characteristics, characteristic coordinates and reduce the equation given below to the canonical form:

$$u_{xx} + (\sec^4 x) u_{yy} = 0. \quad (6)$$

- (b) Transform the following equations to the form:

$$v_{\xi\eta} = cv, \quad c = \text{constant}, \quad u_{xx} - u_{yy} + 3u_x - 2u_y + u = 0$$

by introducing the new variable $v = ue^{-(a\xi + b\eta)}$, where a, b are undetermined coefficients. (6)

- (c) Determine the general solution of the equation given below:

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0 \quad (6)$$

SECTION - III

Attempt any **three** parts out of the following.

4. (a) Determine the solution of the equation:

$$L[u] \equiv u_{xx} + au_x + bu_y + cu = f(x, y)$$

where a, b, c and f are differentiable functions of (x, y) in some domain D^* , under the boundary conditions that u and u_x are prescribed along the curve C in the xy plane. (7)

(b) Solve the Goursat problem

$$xy^3 u_{xx} - x^3 y u_{yy} - y^3 u_x + x^3 u_y = 0, \quad x \neq 0,$$

$$u(x, y) = f(x) \quad \text{on} \quad y^2 + x^2 = 16 \quad \text{for} \quad 0 \leq x \leq 4,$$

$$u(x, y) = g(y) \quad \text{on} \quad x = 0 \quad \text{for} \quad 0 \leq y \leq 4,$$

where $f(0) = g(4)$. (7)

(c) Find the solution of the initial boundary value problem:

$$u_{tt} - c^2 u_{xx} = 0$$

$$u(x, 0) = \sin x, \quad u_t(x, 0) = x^2 \quad (7)$$

(d) Solve the Cauchy problem for the non-homogeneous wave equation

$$u_{tt} = c^2 u_{xx} + e^x,$$

with the initial conditions:

$$u(x, 0) = 5, \quad u_t(x, 0) = x^2 \quad (7)$$

SECTION - IV

Attempt any three parts out of the following.

5. (a) Solve using the method of separation of variables:

$$u_t = k u_{xx}, \quad 0 < x < l, \quad t > 0.$$

$$u(0, t) = 0, \quad t \geq 0$$

$$u(l, t) = 1, \quad t \geq 0$$

$$u(x, 0) = \sin \frac{\pi x}{2l}, \quad 0 \leq x \leq l. \quad (7)$$

(b) Using the method of separation of variables, discuss the problem of vibrating string:

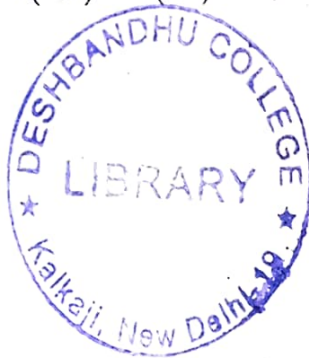
$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \quad 0 < x < l, \quad t > 0, \\ u(x, 0) &= f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq l, \\ u(0, t) &= u(l, t) = 0, \quad t > 0. \end{aligned} \quad (7)$$

(c) Determine the solution of Initial Boundary Value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + x^2, \quad 0 < x < 1, \quad t > 0, \\ u(x, 0) &= x, \quad u_t(x, 0) = 0, \quad 0 \leq x \leq 1, \\ u(0, t) &= 0, \quad u(1, t) = 1, \quad t > 0. \end{aligned} \quad (7)$$

(d) Find the solution of the plucked string of length l equation given by:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ u(x, 0) &= 0, \\ u_t(x, 0) &= \begin{cases} \frac{v_0 x}{a}, & 0 \leq x \leq a \\ \frac{v_0 (l-x)}{(l-a)}, & a \leq x \leq l \end{cases} \\ u(0, t) &= u(l, t) = 0, \quad t > 0. \end{aligned} \quad (7)$$



21

2018

This question paper contains 4 printed pages.

Your Roll No.

S. No. of Paper : 6633
Unique paper code : 32351402
Name of the paper : Reimann Integration and Series of Functions
Name of course : B.Sc. (Hons.) Mathematics
Semester : IV
Duration : 3 hours
Maximum marks : 75

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any two parts from each question.
All questions are compulsory.



Q.1 (a) Prove that a bounded function f on $[a, b]$ is integrable if and only if for each $\varepsilon > 0$, $\exists$ a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. (6)

(b) Show that if f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$, and:

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Hence, show that :

$$\left| \int_{-2\pi}^{2\pi} x^2 \sin^8(e^x) dx \right| \leq \frac{16\pi^3}{3}. \quad (6)$$

(c) Suppose f and g are continuous functions on $[a, b]$, and $g(x) \geq 0 \quad \forall x \in [a, b]$. Prove that, $\exists x \in [a, b]$ such that:

$$\int_a^b f(t)g(t) dt = f(x) \int_a^b g(t) dt. \quad (6)$$

Q.2 (a) (i) The Dirichlet function $f: [0, 1] \rightarrow \mathbf{R}$ is defined by :

$$f(x) = \begin{cases} 1, & x \in [0, 1] \cap \mathbf{Q} \\ 0 & x \in [0, 1] \setminus \mathbf{Q} \end{cases}$$

P. T. O.

Is f Riemann Integrable? Justify. (3)

(ii) Show that a decreasing function on $[a, b]$ is integrable. (3)

(b) (i) State Fundamental Theorem of Calculus-I. (2)

(ii) Let f be an integrable function on $[a, b]$. Let P be a partition of $[a, b]$ and P^* be a refinement of P such that $P^* = P \cup \{u\}$. Show that $L(f, P) \leq L(f, P^*)$. (4)

(c) (i) For a bounded function f on $[a, b]$, define the Riemann Sum associated with a partition P . Hence, give Riemann's definition of integrability. (3)

(ii) Calculate $\lim_{h \rightarrow 0} \frac{1}{h} \int_3^{3+h} e^{t^2} dt$ (3)

3. (a) Show that:

$$\beta(x, y) = \int_0^1 t^x (1-t)^y dt \text{ converges} \Leftrightarrow x > 0, y > 0. \quad (7)$$

(b) (i) Define Improper Integral of Type-II. When does it converge? When is it said to diverge? (5)

(ii) Determine if $\int_{-\infty}^{\infty} e^x dx$ is a convergent integral. Justify. (2)

(c) (i) Determine if the following integrals are Improper Integrals. If so, what kind? Justify.

$$\int_0^{\pi/2} \frac{\sin x}{\sqrt[3]{x}} dx, \quad \int_0^1 x \ln(x) dx \quad (4)$$

(ii) Prove that $\int_{\pi}^{\infty} \frac{\sin x}{x} dx$ converges absolutely. (3)

4. (a) State and prove Cauchy's criterion for uniform convergence for sequences of functions. (6)

(b) (i) Is the sequence $\{f_n\}$ where :

$$f_n = \frac{1}{n} \sin(nx + n).$$

uniformly convergent on $\mathbf{R}$? Justify.

(ii) Suppose a sequence $\{f_n\}$ converges uniformly to f on a set A , and further suppose that each f_n is bounded on A . Show that the limit function f is bounded on A . (2)

(c) Show that if $a > 0$, then the sequence $\{n^2 x^2 e^{-nx}\}$ converges uniformly on the interval $[a, \infty)$, but that it does not converge uniformly on the interval $[0, \infty)$. (4)

5. (a) If f_n is continuous on $D \subseteq \mathbf{R}$, for each $n \in \mathbf{N}$ and if $\sum f_n$ converges to f uniformly on D , then f is continuous on D . (6)

(b) Show that the series of functions $\sum \frac{x^n}{(1+x^n)}$, $x \geq 0$, converges uniformly on $[0, a]$ for $0 < a < 1$, but is not uniformly convergent on $[0, 1)$. (6)

(c) A sequence $\{f_n\}$ converges uniformly to f on A_0 , if for each $\varepsilon > 0$ there is a natural number $K(\varepsilon)$, depending only on ε , such that if $n \geq K(\varepsilon)$, then:

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in A_0.$$

Hence state a necessary and sufficient condition for a sequence $\{f_n\}$ to fail to converge uniformly on A_0 to f .

Apply this condition on sequence $f_n(x) = \frac{x}{n}$, for $x \in \mathbf{R}$ to examine for uniform convergence. (6)

6. (a) (i) Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then show that $\int_0^x f(t) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ for $|x| < R$

(ii) State when is a power series differentiable term by term.

(b) (i) Apply Abel's Theorem to show :

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots \quad (3.5)$$

(ii) Given $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. Evaluate:

$$\sum_{n=1}^{\infty} \frac{n}{5^n}, \quad \sum_{n=1}^{\infty} \frac{n^2 (-1)^n}{3^n} \quad (3)$$

(c) (i) Apply Ratio Test for series to show that radius of convergence R of the power series $\sum a_n x^n$ is given by

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|, \text{ whenever the limit exists.} \quad (3.5)$$

(ii) Find radius of convergence for :

$$\sum_{n=2}^{\infty} (\ln(n))^{-1} x^n, \quad \sum_{n=0}^{\infty} \frac{(n!)^3 x^n}{(3n)!} \quad (3)$$



(22)

This question paper contains 4 printed pages.

Your Roll No. 2415118

S. No. of Paper : 6634

Unique paper code : 32351403

Name of the paper : Ring Theory and Linear Algebra I

Name of course : B.Sc. (Hons.) Mathematics

Semester : IV

Duration : 3 hours

Maximum marks : 75



(Write your Roll No. on the top immediately
on receipt of this question paper.)

Attempt any **three** parts from each question.

All questions are compulsory.

1.(a) Prove that the intersection of any collection of subrings of a ring R is a subring of R . Is union of two subrings again a subring? Justify. 3+2=5

(b) Show that if an element a in $\mathbb{Z}_n$ is a unit, then a and n are relatively prime.

Show that for every prime p , $\mathbb{Z}_p$, the ring of integers modulo p , is a field if and only if p is a prime. 2+3=5

(c) Define characteristic of a ring. Show that the characteristic of an integral domain is either 0 or a prime. 1+4=5

(d) Show that the nilpotent elements of a commutative ring R form a subring of R . Let a be a nilpotent element of R , prove that $1 - a$ has a multiplicative inverse in R . 3+2=5

P. T. O.

2. (a) Show that a maximal ideal is a prime ideal but not conversely. 3+2=5

(b) Let R be a ring with unity 1 and $\phi: R \rightarrow S$ be an onto ring homomorphism, then $\phi(1)$ is the unity of S , where $S \neq \{0\}$. What can you say if ϕ is not onto? Justify. 3+2=5

(c) Determine all ring homomorphisms from $\mathbb{Z}_{20}$ to $\mathbb{Z}_{30}$. 5

(d) Show that a field contains $\mathbb{Z}_p$ or $\mathbb{Q}$, where p is a prime. 5

3. (a) Prove that sum of two subspaces W_1 and W_2 of a vector space V is again a subspace of V . Further, show that $W_1 + W_2$ is the smallest subspace of V containing both W_1 and W_2 . 5

(b) Show that for any subset S of a vector space V , $\text{span}(S)$ is the smallest subspace of V containing S . 5

(c) Check whether $S = \{(0, 1, -2), (1, -1, 1), (1, 2, 1)\}$ forms a basis of $\mathbb{R}^3$ or not. 5

(d) Let V be a vector space over a field F and let S_1, S_2 be two subsets of V such that $S_1 \subseteq S_2 \subseteq V$. Then prove that:

(i) If S_1 is linearly dependent, then S_2 is also linearly dependent.

(ii) If S_2 is linearly independent, then so is S_1 . 5

4. (a) Let V be a finite-dimensional vector space and if W be a subspace of V , then prove that any basis for W can be extended to a basis for V . 5

(b) Suppose that W be a subspace of a finite-dimensional vector space V , then W is finite-dimensional and $\dim(W) \leq \dim(V)$.

Further, if $\dim(W) = \dim(V)$, then $V = W$.

(c) Prove that the set of solutions to the system of linear equations :

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_1 - 3x_2 + x_3 = 0$$

is a subspace of $\mathbb{R}^3$. Find a basis for this subspace. 5

(d) Extend the set $S = \{(1, 0, 0, 0), (1, 1, 0, 0), (0, 0, 1, 1)\}$ to form a basis of $\mathbb{R}^4$. 5

5. (a) Let V and W be vector spaces and $T : V \rightarrow W$ be a linear transformation. Then prove that $N(T)$ and range $R(T)$ are subspaces of V and W , respectively. 5

(b) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by:

$$T(x_1, x_2) = (2x_1 - x_2, 3x_1 + 4x_2, x_1).$$

Let β be the standard ordered basis for $\mathbb{R}^2$ and :

$$\gamma = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}.$$

Compute $[T]_{\beta}^{\gamma}$. 5

(c) Let V be a vector space and let $T : V \rightarrow V$ be a linear transformation. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$, where $T_0(v) = 0$, for all $v \in V$. 5

(d) Let :

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } \beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

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be an ordered basis of $\mathbb{R}^3$. Then, find $[L_A]_\beta$ and also find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$, where L_A is a left-multiplication transformation.

